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On the force transmissibility of a vibration isolator with quasi-zero-stiffness

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Abstract

In this article the force transmissibility of a quasi-zero-stiffness (QZS) isolator is considered. The isolator comprises a vertical spring and two oblique springs that are either linear, linear with pre-stress or softening nonlinear with pre-stress. The force transmissibility of such a system is derived and compared with that of a linear system. Assuming light damping, simple approximate expressions for the maximum transmissibility and jump-down frequencies are derived. It is shown that there are advantages in having nonlinear and pre-stressed oblique springs compared to the other configurations, and that all the QZS systems can outperform the linear system provided that the system parameters are chosen appropriately.

1. Introduction

The use of passive isolators is ubiquitous in engineering systems [1,2]. In the simplest case when the isolator is linear, a low natural frequency, which is desirable, can only be achieved by having a large static deflection, which is undesirable. This disadvantage can be overcome by adding oblique springs in order to obtain a high static stiffness, small static displacement, small dynamic stiffness, and hence low natural frequency [2,3]. By choosing appropriate stiffness and geometry for the oblique springs zero dynamic stiffness can be achieved, i.e. a so-called quasi-zero stiffness (QZS) isolator can be realised. Sometimes this type of system is called an ultra-low frequency vibration isolator [4]. A static analysis of a simple model of a QZS isolator has been presented by Carrella et al. [5]. Kovacic et al. [6] have extended the study considering linear and nonlinear pre-stressed oblique springs.

In this article, a dynamic analysis of these QZS systems is performed and an expression for the force transmissibility derived using the harmonic balance (HB) method in a similar way to Ravindra and Mallik [7], who also investigated the performance of nonlinear isolators. Approximate expressions for the maximum transmissibility and jump-down frequency of the QZS isolator are derived and the performance of the isolator is compared with that of a linear vibration isolator.

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2. Static analysis

The QZS isolator considered is shown schematically in Fig. 1. The system consists of a vertical spring connected at point O with two oblique springs. The vertical spring is of stiffness k_2. Three different configurations regarding the characteristics of the oblique springs are considered. First, the case when they are linear with stiffness k_1. Second, the case when the two oblique springs are linear with the same stiffness but pre-stressed, i.e. compressed by length δ. The third configuration corresponds to the case when the oblique springs have a softening characteristic (i.e. are nonlinear) with the restoring force expressed by the cubic polynomial $k_1 x - k_3 x^3$ and are also pre-stressed. In the study presented herein, these configurations and the corresponding parameters are labelled I, II and III, respectively, as shown in Fig. 1. The geometry of the system is defined by the parameters a and h, while the coordinate x defines the displacement from the initial *unloaded* position. A comprehensive static analysis of the different QZS isolator configurations system is given in [5,6], and so only an overview of the analysis is given here. In the most general case, which relates to the geometrically and physically nonlinear system, Configuration III, the relationship between the vertical applied force f and displacement x is derived in Appendix A and is given by

$$f = k_2 x + 2k_1(h-x)\left(\frac{\sqrt{a^2+h^2}+\delta}{\sqrt{a^2+(h-x)^2}} - 1\right) + 2k_3 \frac{h-x}{\sqrt{a^2+(h-x)^2}})\sqrt{a^2+(h-x)^2} - \sqrt{a^2+h^2} - \delta)^3, \quad (1)$$

which can be written in non-dimensional form as

$$\hat{f} = \hat{x} + 2\alpha(\sqrt{1-\hat{a}^2} - \hat{x})\left(\frac{\hat{\delta}+1}{\sqrt{\hat{x}^2 - 2\sqrt{1-\hat{a}^2}\hat{x}+1}} - 1\right) - 2\beta(\sqrt{1-\hat{a}^2} - \hat{x})(\hat{x}^2 - 2\sqrt{1-\hat{a}^2}\hat{x}+1) \times \left(\frac{\hat{\delta}+1}{\sqrt{\hat{x}^2 - 2\sqrt{1-\hat{a}^2}\hat{x}+1}} - 1\right)^3, \quad (2)$$

where $\hat{f} = f/(k_2\sqrt{a^2+h^2})$, $\alpha = (k_1/k_2)$, $\hat{a} = a/\sqrt{a^2+h^2}$, $\hat{x} = x/\sqrt{a^2+h^2}$, $\hat{\delta} = \delta/\sqrt{a^2+h^2}$ and $\beta = (k_3(a^2+h^2))/k_2$.

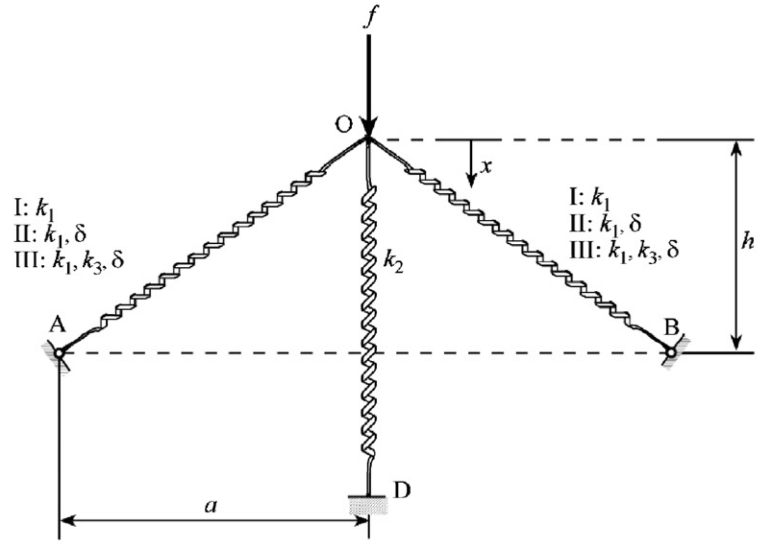

Fig. 1. A three-spring model of a QZS mechanism. The oblique springs are either I linear, II linear with pre-stress, or III nonlinear with cubic softening nonlinearity and with pre-stress. The vertical spring is linear.

Differentiating Eq. (2) with respect to $\hat{x}$ gives the non-dimensional stiffness of the system (normalised by k_2) as

$$\hat{K} = 1 + 2\alpha\left(1 - \hat{a}^2\frac{\hat{\delta}+1}{\sqrt{(\hat{x}^2 - 2\sqrt{1-\hat{a}^2}\hat{x}+1)^3}}\right) - 2\beta\left(\frac{\hat{\delta}+1}{\sqrt{\hat{x}^2 - 2\sqrt{1-\hat{a}^2}\hat{x}+1}} - 1\right)^2$$
$$\times\left[\left(1 - \frac{\hat{\delta}+1}{\sqrt{\hat{x}^2 - 2\sqrt{1-\hat{a}^2}\hat{x}+1}}\right)(3(\sqrt{1-\hat{a}^2}-\hat{x})^2+\hat{a}^2) + 3(\sqrt{1-\hat{a}^2}-\hat{x})^2\frac{\hat{\delta}+1}{\sqrt{\hat{x}^2 - 2\sqrt{1-\hat{a}^2}\hat{x}+1}}\right]. \tag{3}$$

The stiffnesses and geometry are chosen such that when the supported mass is placed on the isolator the vertical spring compresses and the oblique springs become horizontal as shown in Fig. 2. The static equilibrium position is $\hat{x} = \hat{x}_e$, where $\hat{x}_e = \sqrt{1-\hat{a}^2}$. The stiffness of the system at the static equilibrium position is zero provided that

$$\beta = \frac{\alpha}{[1-\hat{a}+\hat{\delta}]^2} - \frac{\hat{a}}{2[1-\hat{a}+\hat{\delta}]^3}. \tag{4}$$

As discussed in [5,6], in addition to the isolator having a QZS characteristic it is desirable for it to have a wide-range of non-dimensional displacements from the static equilibrium position for which the non-dimensional stiffness is less than a prescribed low value $\hat{K}_p$. The non-dimensional stiffness is thus $\hat{K}_p$ at a displacement of $\hat{x}_e \pm \hat{d}$. Substituting for $\hat{x}_{\hat{K}=\hat{K}_p} = \hat{x}_e \pm \hat{d}$ into Eq. (3) gives

$$6\beta\hat{z}^5 - 12(\hat{\delta}+1)\beta\hat{z}^4 + [\hat{K}_p - 1 - 2\alpha + 6(\hat{\delta}+1)^2\beta - 4\hat{a}^2\beta]\hat{z}^3 + 6\gamma^2(\hat{\delta}+1)\beta\hat{z}^2 + 2\alpha\hat{a}^2(\hat{\delta}+1) - 2\gamma^2(\hat{\delta}+1)^3\beta = 0, \tag{5}$$

where $\hat{z}^2 = \hat{d}^2 + \hat{a}^2$. Carrying out numerical optimisation, Eq. (5) can be solved to give, for the case when the oblique springs are pre-stressed and nonlinear, $\hat{a} = 0.5$, $\alpha = 0.51$, and $\hat{\delta} = 0.89$ (for which $\beta = 0.1709$) [6]. The optimisation criteria includes the achievement of the largest displacement from the static equilibrium position at which the prescribed stiffness is equal to that of the vertical spring alone, i.e. $\hat{K}_p = 1$, the condition that the stiffness should never be negative, and the requirement that the stiffness only changes slightly in the neighbourhood of the equilibrium position (the tolerance of $\Delta\hat{K} = 0.0025$ for $\Delta\hat{y} = 0.01$ was introduced, where $\hat{y} = \hat{x} - \hat{x}_e$).

When the oblique springs are linear and unstressed then $\beta = 0$, $\hat{\delta} = 0$ and Eq. (4) becomes

$$\alpha_{\mathrm{I}} = \frac{\hat{a}}{2(1-\hat{a})}. \tag{6}$$

In the case when the oblique springs are linear but pre-stressed then $\beta = 0$ and Eq. (4) becomes

$$\alpha_{\mathrm{II}} = \frac{\hat{a}}{2(1+\hat{\delta}-\hat{a})}. \tag{7}$$

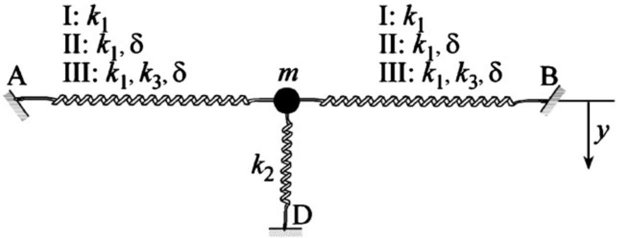

Fig. 2. A QZS isolator loaded with a mass so that it is in the static equilibrium position.

Imposing the condition that the desired stiffness never exceeds that of the vertical spring within the range $\hat{y} = \pm\hat{x}_e$, i.e. $\hat{K}_p = 1$, Eq. (5) gives

$$\hat{a}_{\mathrm{II}} = \frac{1}{\sqrt{1+\hat{\delta}}}. \tag{8}$$

For the case when all three springs are linear, then $\hat{a} = (\frac{2}{3})^{3/2}$ [5], and α is calculated using Eq. (6). When the oblique springs are linear but pre-stressed the pre-stress is arbitrarily chosen so that $\hat{\delta} = 0.5$ [6]. In this case, Eqs. (7) and (8) can be used to calculate α and $\hat{a}$, respectively. The optimum parameters for all three combinations are presented in Table 1 for convenience. The stiffness of the system for the three conditions described above is plotted with respect to the coordinate system $\hat{y}$ in Fig. 3. The circles in Fig. 3 denote the stiffness given by Eq. (3) calculated at $\hat{y} = \pm\hat{x}_e$. Note that because the unloaded position is different in cases I, II and III, then $\hat{x}_e$ is also different in all three cases. However, in the static equilibrium position shown in Fig. 2, the springs that were initially oblique, are always horizontal.

It can be seen from Fig. 3 that for all three cases, although they have zero stiffness at the static equilibrium position, the characteristics of the oblique springs govern the stiffness of the isolator away from this position. When the oblique springs are linear (case I), the stiffness of the system increases rapidly as the displacement increases from the static equilibrium position. The stiffness for large excursions from this position is greater than that of the vertical spring alone (in the limit it tends to the sum of the vertical and the oblique springs). Of course this is undesirable, as it results in a hardening system, which (as shown later) can increase the minimum frequency at which vibration isolation can occur. By including some pre-stress into the oblique springs (case II), there is less of a hardening effect, which is desirable as it will have a beneficial effect on vibration isolation, and finally by making the oblique springs nonlinear with a softening characteristic (case III), the hardening effect is reduced even further. Thus, theoretically at least, the three cases considered could be ranked as III, II then I in descending order of merit. It may prove difficult, however, to fabricate the oblique springs with precise nonlinear characteristics. Such issues are not discussed here, as they are outside the scope of the paper.

3. Dynamic response of the QZS system

To include the influence of damping in the isolator, a linear viscous damper, with damping coefficient c_2, is added in parallel with the vertical spring. The equivalent system of the isolator and the supported mass, in which the optimised isolator stiffness is denoted by K_{QZS}, is shown in Fig. 4. Provided that the displacement about the static equilibrium position is small, the restoring force, given by Eq. (2), can be expanded using the Maclaurin series up to third order. If it is also assumed that the system is optimised such that the system has

Table 1
Expressions for the optimum values of the parameters used in the simulations, and the optimum values of the nonlinear parameter, γ.

Configuration	Spring types	Optimum parameters	Optimum nonlinear parameter, γ
I	Linear oblique springs	$\hat{a}_{\mathrm{I}} = (\frac{2}{3})^{3/2}$	$\gamma_{\mathrm{I}} = \frac{1}{2\hat{a}^2(1-\hat{a})} = 3.7033$
II	Linear oblique springs with pre-stress	$\hat{a}_{\mathrm{II}} = (\frac{2}{3})^{1/2}$ $\hat{\delta}_{\mathrm{II}} = 0.5$	$\gamma_{\mathrm{II}} = \frac{1}{2\hat{a}^2(1-\hat{a}^3)} = 1.6459$
III	Softening nonlinear oblique springs with pre-stress	$\hat{a}_{\mathrm{III}} = 0.5$ $\alpha_{\mathrm{III}} = 0.51$ $\beta = 0.1709$ $\hat{\delta}_{\mathrm{III}} = 0.89$	$\gamma_{\mathrm{III}} = -2\beta + 3\beta\frac{1+\hat{\delta}}{\hat{a}} + \alpha\frac{1+\hat{\delta}}{\hat{a}^3} - \beta\frac{(1+\hat{\delta})^3}{\hat{a}^3} = 0.0783$

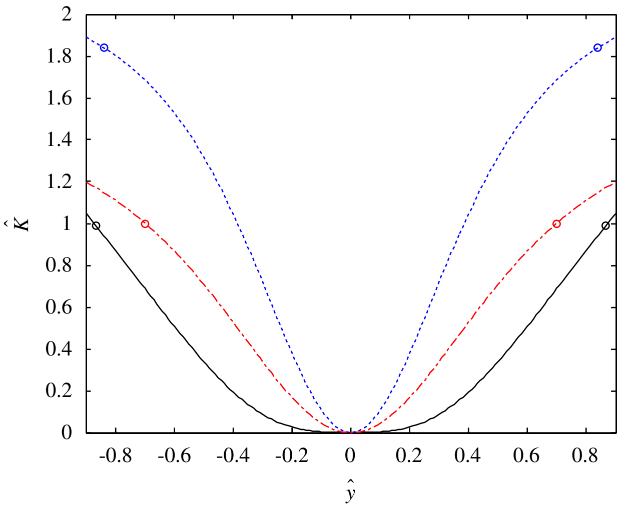

Fig. 3. Non-dimensional stiffness of the QZS system with the linear oblique springs (dashed line), linear pre-stressed oblique springs (dashed-dotted line) and nonlinear pre-stressed springs (solid line). The circles denote the stiffness at $\hat{y} = \pm\hat{x}_e$.

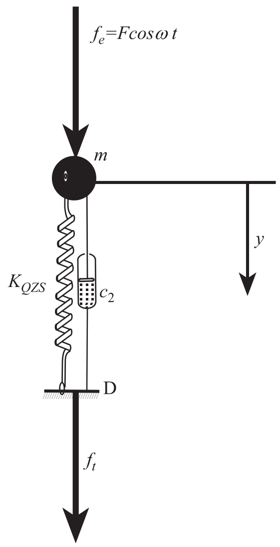

Fig. 4. Equivalent sdof system of the isolator supporting a mass m.

zero stiffness at the static equilibrium position, (Eq. (4) applies), the equation of motion of the system about the static equilibrium position can be approximated by Duffing's equation with no linear term. For harmonic excitation, the non-dimensional equation of motion is given by [6]

$$\ddot{\hat{y}} + 2\zeta\dot{\hat{y}} + \gamma\hat{y}^3 = \hat{f}_e, \tag{9}$$

where

$$\omega_0^2 = \frac{k_2}{m}, \quad \tau = \omega_0 t, \quad \zeta = \frac{c_2\omega_0}{2k_2}, \quad \Omega = \frac{\omega}{\omega_0}, \quad \hat{f}_e = \hat{F}\cos\Omega\tau, \quad \hat{F} = \frac{F}{k_2\sqrt{a^2+h^2}},$$

and the nonlinear coefficient γ depends on the type of configuration. For Configuration III γ is given by

$$\gamma_{\mathrm{III}} = -2\beta + 3\beta\frac{1+\hat{\delta}}{\hat{a}} + \alpha\frac{1+\hat{\delta}}{\hat{a}^3} - \beta\frac{(1+\hat{\delta})^3}{\hat{a}^3}. \tag{10}$$

In the case when the oblique springs are linear but pre-stressed the nonlinear coefficient γ is determined by setting $\beta = 0$ and using Eqs. (7) and (8) to give

$$\gamma_{\mathrm{II}} = \frac{1}{2\hat{a}^2(1-\hat{a}^3)}. \tag{11}$$

In the case when all the springs are linear, then the nonlinear coefficient γ is determined by setting $\beta = 0$ and $\delta = 0$, and using Eq. (6) to give

$$\gamma_{\mathrm{I}} = \frac{1}{2\hat{a}^2(1-\hat{a})}. \tag{12}$$

Expressions for the optimum values of the non-dimensional cubic coefficient of the restoring force for the different configurations are also given in Table 1. To investigate the dynamic behaviour of the QZS system the HB method is used to determine its approximate response at the excitation frequency. The reasons for this choice are its simplicity and applicability to strongly nonlinear systems [8–10]. In this study, attention is restricted to the system parameters for which the response has predominantly the same frequency as the harmonic excitation, so that all other Fourier components can be neglected. The solution to Eq. (9) is thus assumed to be $\hat{y} = \hat{Y}\cos(\Omega\tau + \varphi)$ which yields

$$\hat{Y}^2\Omega^4 + (4\zeta^2\hat{Y}^2 - \tfrac{3}{2}\gamma\hat{Y}^4)\Omega^2 + \tfrac{9}{16}\gamma^2\hat{Y}^6 - \hat{F}^2 = 0. \tag{13}$$

This is a quadratic equation in Ω^2 which can be solved to give

$$\Omega_{1,2} = \frac{1}{2}\sqrt{3\gamma\hat{Y}^2 - 8\zeta^2 \pm \frac{4}{\hat{Y}}\sqrt{4\zeta^4\hat{Y}^2 - 3\gamma\hat{Y}^4\zeta^2 + \hat{F}^2}}, \tag{14}$$

which are the resonant and non-resonant branches in the frequency response function. For small values of damping when $\zeta \ll 1$, the jump-down frequency occurs approximately when $\hat{Y}$ is a maximum. This maximum can be determined by noting that it occurs when $\Omega_{1,2}$ are equal, which is when $4\zeta^4\hat{Y}^2 - 3\gamma\hat{Y}^4\zeta^2 + \hat{F}^2 = 0$ in Eq. (14). Thus, the maximum value of $\hat{Y}$ is found to be

$$\hat{Y}_{\max} = \sqrt{\frac{2\zeta^3 + \sqrt{4\zeta^6 + 3\gamma\hat{F}^2}}{3\zeta\gamma}}. \tag{15a}$$

If $(4/3)(\zeta^6/\gamma\hat{F}^2) \ll 1$ Eq. (15a) approximates to

$$\hat{Y}_{\max} \approx \sqrt[4]{\frac{\hat{F}^2}{3\zeta^2\gamma}}. \tag{15b}$$

Substituting Eq. (15a) into Eq. (14) gives the jump-down frequency

$$\Omega_d = \frac{1}{\sqrt{2\zeta}}\sqrt{\sqrt{\zeta^6 + \frac{3}{4}\gamma\hat{F}^2} - 3\zeta^3}. \tag{16a}$$

If $4(\zeta^6/\gamma\hat{F}^2) \ll 1$ Eq. (16a) approximates to

$$\Omega_d \approx \sqrt[4]{\frac{3\gamma\hat{F}^2}{16\zeta^2}}. \tag{16b}$$

To validate the approximate expressions for the maximum value of $\hat{Y}$ given by Eq. (15b) and the jump-down frequency given in Eq. (16b), a comparison with the values calculated numerically is conducted. The numerical values are obtained by solving the differential equation of motion in Eq. (9) using direct numerical integration. The comparison made for $\zeta = 0.01$ and different values of $\hat{F}$ is given in Table 2. Reasonably good agreement between the numerical and analytical results can be seen.

Table 2
Comparison of the analytical results for the maximum value of $\hat{Y}$ Eq. (15b) and the jump down frequency Eq. (16b) with the numerically obtained results for $\zeta = 0.01$ and different values of $\hat{F}$, and an absolute error of the analytical results with respect to the numerical ones.

Configuration	$\hat{Y}_{\max}$			Ω_d		
	Analytical value	Numerical value	Absolute error (%)	Analytical value	Numerical value	Absolute error (%)
$\hat{F} = 0.001$						
I	0.1732	0.1714	1.05	0.2887	0.2782	3.77
II	0.2121	0.2060	2.96	0.2357	0.2298	2.57
III	0.4542	0.4554	0.26	0.1101	0.1105	0.36
$\hat{F} = 0.004$						
I	0.3464	0.3433	0.9	0.5773	0.5750	0.4
II	0.4243	0.4256	0.31	0.4714	0.4752	0.8
III	0.9085	0.9079	0.07	0.2282	0.2249	1.47
$\hat{F} = 0.006$						
I	0.4243	0.4231	0.28	0.7071	0.7023	0.68
II	0.5196	0.5116	1.56	0.5773	0.5605	3
III	1.1126	1.1140	0.13	0.2696	0.2702	0.22
$\hat{F} = 0.01$						
I	0.5477	0.5426	0.94	0.9128	0.9149	0.23
II	0.6708	0.6637	1.07	0.7453	0.7458	0.07
III	1.4364	1.4328	0.25	0.3481	0.3487	0.17

Of interest in this paper is the amplitude of the force transmissibility, which is defined as the ratio of the magnitude of the force transmitted to the rigid foundation, to the magnitude of the excitation force. It is given by

$$|T| = \frac{\hat{F}_t}{\hat{F}}. \tag{17}$$

The non-dimensional force transmitted through the nonlinear spring and the dashpot that comprises the isolator, which is shown in Fig. 4, is given by

$$\hat{f}_t = 2\zeta\dot{\hat{y}} + \gamma\hat{y}^3. \tag{18}$$

Using the HB method the component of the non-dimensional transmitted force at the excitation frequency has the form $\hat{f}_t = \hat{F}_t \cos(\Omega\tau + \varphi_t)$, where the magnitude of the force is given by

$$\hat{F}_t = \sqrt{(\tfrac{3}{4}\gamma\hat{Y}^3)^2 + (2\zeta\Omega\hat{Y})^2}. \tag{19}$$

Thus the magnitude of the transmissibility can be determined by using the two solutions for $\Omega_{1,2}$ given in Eq. (14) to give

$$|T|_1 = \sqrt{\frac{\frac{9}{16}\gamma^2\hat{Y}^6 + 4\zeta^2\Omega_1^2\hat{Y}^2}{\hat{F}^2}}, \tag{20a}$$

and

$$|T|_2 = \sqrt{\frac{\frac{9}{16}\gamma^2\hat{Y}^6 + 4\zeta^2\Omega_2^2\hat{Y}^2}{\hat{F}^2}}, \tag{20b}$$

To determine an approximate expression for the peak transmissibility either $|T|_1$ or $|T|_2$ can be used. Provided that damping is small, and the frequency is close to that when the transmissibility is maximum, then the first term in Eq. (20a) will be much larger than the second term, i.e. $(16\zeta^3/\sqrt{3\gamma^3})\hat{F} \ll 1$, so that $|T|_1 \approx 3\gamma\hat{Y}^3/4\hat{F}$. Substituting for the maximum value of the response given by Eq. (15b) gives an approximate expression for the maximum transmissibility

$$|T|_{\max} \approx \sqrt{\frac{\sqrt{3\gamma}\hat{F}}{16\zeta^3}}, \tag{21}$$

which occurs at the jump-down frequency given by Eq. (16b). In the derivation of Eq. (21), three conditions were imposed: the first condition leading to Eq. (15b), which can be written as $\zeta^6 \ll \frac{3}{4}\gamma F^2$, the second condition leading to Eq. (16b), which can be expressed as $\zeta^6 \ll \frac{1}{4}\gamma F^2$ and finally the condition imposed just prior to Eq (21), which can be written as $\zeta^6 \ll (\frac{3}{256}\gamma^2)\gamma F^2$. Given the values of γ used in this paper, the third condition is the strictest condition and should be used when determining the validity of the approximate expression for $|T|_{\max}$ given by Eq. (21).

At frequencies when $\Omega \gg 1$, and if it is assumed that the jump-down has occurred, the displacement response of the system can be calculated using Eq. (13). Well above the frequency where the jump-down occurs, damping has little influence on the response, so it can be set to zero. Also it can be assumed that $\hat{Y} \ll 1$ so that

$$\hat{Y} \approx \frac{\hat{F}}{\Omega_2^2}. \tag{22}$$

If it is assumed that the second term in the expression for the transmissibility well above the jump-down frequency given by Eq. (20b) is much larger than the first term, then the first term can be neglected so that $|T|_2 \approx 2\zeta\Omega_2\hat{Y}/\hat{F}$. Substituting for $\hat{Y}$ from Eq. (22) and letting $\Omega = \Omega_2$ because $\Omega \gg \Omega_d$ then

$$|T|_2 \approx \frac{2\zeta}{\Omega}. \tag{23}$$

4. Comparison between the linear and QZS systems

In a linear system, the lowest frequency at which vibration isolation occurs is $\sqrt{2} \times$ natural frequency. However, in a hardening nonlinear system isolation occurs after the jump-down frequency. Also of importance, however, is the peak force transmissibility and this is compared for the linear and nonlinear system in this section.

The transmissibility of the linear system is given by [1]

$$|T|_{\text{lin}} = \sqrt{\frac{1 + 4\zeta^2\Omega^2}{(1 - \Omega^2)^2 + 4\zeta^2\Omega^2}}, \tag{24a}$$

and the maximum amplitude of the lightly damped linear system is given approximately by

$$|T|_{\text{lin(max)}} = \frac{1}{2\zeta}. \tag{24b}$$

Setting $\Omega_d = 1$ in Eq. (16b), and rearranging for $\hat{F}^2$ gives $\hat{F}^2 = 16\zeta^2/3\gamma$, which can be substituted into Eq. (21) to give an identical expression to that in Eq. (24b). This shows that when the parameters of the nonlinear system are adjusted so that the jump-down frequency occurs at the undamped natural frequency of the linear system, the maximum amplitude of the transmissibility is the same as that for the linear system. This is illustrated in Fig. 5, where the transmissibility of the nonlinear system is plotted using Eqs. (20a,b). If the excitation force is increased and the system parameters kept the same, then the peak in the transmissibility is greater and occurs at a higher frequency.

The transmissibility of the QZS system for three values of the nonlinear parameter γ corresponding to those given in Table 1 is plotted in Fig. 6 where the parameters are $\zeta = 0.01$ and $\hat{F} = 0.01$. The transmissibility of a

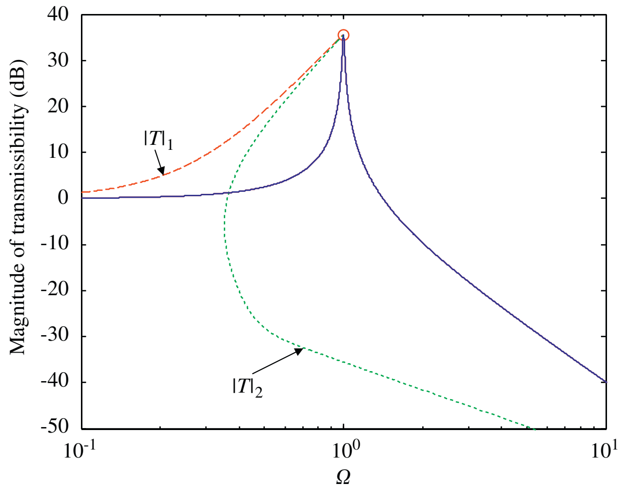

Fig. 5. Transmissibility of the linear system (solid line). Transmissibility of the QZS system with Configuration III ($\gamma = 3.7033$) and the non-dimensional excitation force, $\hat{F} = 0.01$. The damping ratio ζ, has been chosen so that $3\gamma\hat{F}^2/16\zeta^2 = 1$ (dashed line and dotted line). The circle is given by $|T|_{\max} = 1/2\zeta$.

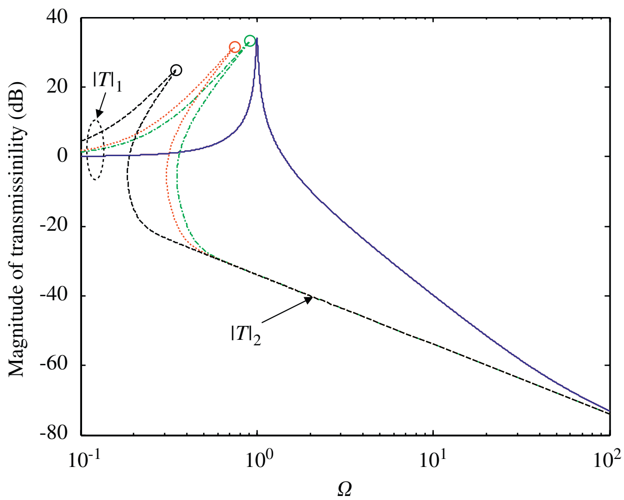

Fig. 6. Transmissibility of the optimised QZS system: Configuration I (dashed-dotted line); Configuration II (dotted line); Configuration III (dashed line); Transmissibility of a linear system (solid line). The circles are the approximate maximum values of the transmissibility given by Eq. (21) at the approximate jump-down frequency given by Eq. (16b). The damping ratio, $\zeta = 0.01$ and the non-dimensional excitation force, $\hat{F} = 0.01$.

linear system given by Eq. (24a) is also plotted; it is formed by simply removing the oblique springs from the original three-spring system. It can be seen that, for the values of the parameters used, all the QZS systems outperform the linear system in that the jump-down frequencies are lower than the natural frequency of the linear system. Moreover, the maximum amplitudes of the transmissibility for all the QZS systems are less than that of the linear system. It can also be seen that there are clear advantages in making the oblique springs both nonlinear with softening characteristic and pre-stressed. At high frequencies, much greater than either the natural frequency of the linear system or the jump-down frequency of the QZS system, the transmissibilities of both systems are given by Eq. (23), which shows that they reduce at 20 dB per decade. This can also be seen in Fig. 6.

By comparing the approximate maximum amplitude of the transmissibility of the linear system with that of the QZS system given by Eq. (21), the following inequality for the amplitude of the excitation force can be determined such that the maximum amplitude is less than that of the linear system

$$\hat{F} < \frac{4\zeta}{\sqrt{3\gamma}}. \tag{25}$$

If this inequality holds, then the QZS system will always outperform the linear system; the jump-down frequency and the maximum transmissibility will always be less than the natural frequency and the maximum amplitude of the transmissibility of the linear system, respectively. If the amplitude of the excitation force is such that the inequality given in Eq. (25) does not hold then the nonlinear system does not outperform the linear system.

5. Conclusions

In this paper the transmissibility of three configurations of a QZS system have been investigated and compared with that of a linear system. The QZS systems were all formed with a vertical spring and were connected to oblique springs that were either linear, linear and pre-stressed or nonlinear and pre-stressed. Assuming light damping, approximate expressions for the maximum transmissibility and jump-down frequencies were derived. It was shown that there are advantages in having nonlinear and pre-stressed oblique springs, and that all the QZS systems can outperform the linear system provided that the system parameters are chosen appropriately. At frequencies much greater than the jump-down frequency the QZS systems behave as the linear isolator with a transmissibility reducing at a rate of 20 dB per decade.

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Appendix A. Derivation of the expression for the static force–displacement characteristic

For the vibration isolator shown in Fig. 1 the relationship between the applied force f and the resulting displacement x can be found by means of the principal of virtual work. This requires the total work by the force f, and the reactions of the oblique springs in the x direction f_{1x}, and the vertical spring f_2, to be zero for a virtual displacement δx, i.e.,

$$(f + 2f_{1x} + f_2)\delta x \equiv 0. \tag{A.1}$$

Both oblique springs are assumed to be softening, with linear stiffness k_1 and cubic softening nonlinear stiffness coefficient k_3 for a single oblique spring. In addition, they are pre-stressed, i.e. compressed by length δ before connecting with the vertical spring. The corresponding restoring force f_1 is given by

$$f_1 = k_1\left(\sqrt{a^2 + (h-x)^2} - \sqrt{a^2 + h^2} - \delta\right) - k_3\left(\sqrt{a^2 + (h-x)^2} - \sqrt{a^2 + h^2} - \delta\right)^3. \tag{A.2}$$

Its scalar component in the x direction is

$$f_{1x} = f_1 \frac{h-x}{\sqrt{a^2 + (h-x)^2}}. \tag{A.3}$$

The reaction of the vertical unstressed spring of stiffness k_2 is

$$f_2 = -k_2 x. \tag{A.4}$$

Combining (A.1)–(A.4) gives

$$f = k_2 x + 2k_1(h - x)\left(\frac{\sqrt{a^2 + h^2} + \delta}{\sqrt{a^2 + (h - x)^2}} - 1\right) + 2k_3 \frac{h - x}{\sqrt{a^2 + (h - x)^2}}\left(\sqrt{a^2 + (h - x)^2} - \sqrt{a^2 + h^2} - \delta\right)^3. \quad (A.5)$$

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